Model Counting with Algebraic Decision Diagrams

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Joint work with Moshe Vardi and Jeffrey Dudek

Overview: Model Counting

Satisfiability problem (SAT): whether Boolean formula has satisfying assignment

- Complexity: NP-complete [Cook, 1971]

Model counting problem (#SAT): number of satisfying assignments of Boolean formula

- Complexity: #P-complete [Valiant, 1979]
- Applications:
  - Hardware verification [Naveh et al., 2007]
  - Bayesian inference [Sang et al., 2005]
    - Medical diagnosis [Shwe et al., 1991]

Contents

Progress

1. Model Counting and Bayesian Inference
2. Algebraic Decision Diagrams for Model Counting
3. Empirical Evaluation

Boolean Logic Syntax

Boolean formula (in Conjunctive Normal Form):

$$\varphi = x_1 \land \neg x_2 \land (x_1 \lor \neg x_2) \land (x_1 \lor x_2 \lor \neg x_3)$$

- Boolean variables: x_1, x_2, x_3
- Positive literals: x_1, x_2
- Negative literals: $\neg x_2, \neg x_3$

Boolean Logic Syntax

Boolean formula (in Conjunctive Normal Form):

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$

- Boolean variables: x_1, x_2, x_3
- Positive literals: x_1, x_2
- Negative literals: $\neg x_2, \neg x_3$
- Disjunctions (clauses): $x_1, \neg x_2, (x_1 \vee \neg x_2), (x_1 \vee x_2 \vee \neg x_3)$

Boolean Logic Syntax

Boolean formula (in Conjunctive Normal Form):

$$\varphi = x_1 \land \neg x_2 \land (x_1 \lor \neg x_2) \land (x_1 \lor x_2 \lor \neg x_3)$$

- Boolean variables: x_1, x_2, x_3
- Positive literals: x_1, x_2
- Negative literals: $\neg x_2, \neg x_3$
- Disjunctions (clauses): $x_1, \neg x_2, (x_1 \lor \neg x_2), (x_1 \lor x_2 \lor \neg x_3)$
- Conjunction (formula): $x_1 \land \neg x_2 \land (x_1 \lor \neg x_2) \land (x_1 \lor x_2 \lor \neg x_3)$

Model Counting Problem [Valiant, 1979]

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$
$$\varphi : \mathbb{B}^3 \to \mathbb{B}$$

x_1	x_2	x_3	$\varphi(x_1, x_2, x_3)$	Satisfying assignment?
T	T	T	F	No
T	T	F	F	No
T	F	T	**T**	**Yes**, model $M_1 = \{x_1, \neg x_2, x_3\}$
T	F	F	**T**	**Yes**, model $M_2 = \{x_1, \neg x_2, \neg x_3\}$
F	T	T	F	No
F	T	F	F	No
F	F	T	F	No
F	F	F	F	No

Model count:

$$\#\varphi = \mathbf{2}$$

Weighted Model Counting Problem

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$

Literal weights	
$W(x_1) = 1.0$	$W(\neg x_1) = 1.0$
$W(x_2) = 0.2$	$W(\neg x_2) = 0.8$
$W(x_3) = 0.3$	$W(\neg x_3) = 0.7$

Model weights:

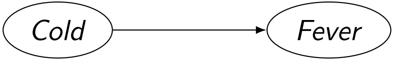
$$\begin{aligned} W(M_1) = W(\{x_1, \neg x_2, x_3\}) &= W(x_1) \cdot W(\neg x_2) \cdot W(x_3) &= 1.0 \cdot 0.8 \cdot 0.3 = 0.24 \\ W(M_2) = W(\{x_1, \neg x_2, \neg x_3\}) &= W(x_1) \cdot W(\neg x_2) \cdot W(\neg x_3) &= 1.0 \cdot 0.8 \cdot 0.7 = 0.56 \end{aligned}$$

Weighted model count:

$$W(\varphi) = W(M_1) + W(M_2) = 0.24 + 0.56 = 0.8$$

Bayesian Network [Pearl, 1985]

Bayesian network:



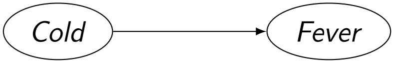
$$P(Cold) = 0.4$$

$$P(Fever|Cold) = 0.3$$
$$P(Fever|\neg Cold) = 0.1$$

Query:

$$P(Cold|Fever) = \ ?$$

Bayesian Network: Reduction to Model Counting [Sang et al., 2005]



$P(Cold) = 0.4$

$P(Fever|Cold) = 0.3$

$P(Fever|\neg Cold) = 0.1$

Literal weights	
$W(x_1) = 0.4$	$W(\neg x_1) = 0.6$
$W(x_2) = 0.3$	$W(\neg x_2) = 0.7$
$W(x_3) = 0.1$	$W(\neg x_3) = 0.9$
$W(x_4) = 1.0$	$W(\neg x_4) = 1.0$

$$\begin{aligned} \varphi = \; & (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee \neg x_4) \wedge \\ & (x_1 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee x_3 \vee \neg x_4) \wedge x_4 \\ \gamma = \; & \varphi \wedge x_1 \end{aligned}$$

Computing answer to query:

$$P(Cold|Fever) = \frac{P(Fever, Cold)}{P(Fever)} = \frac{W(\gamma)}{W(\varphi)} = \frac{0.12}{0.18} = 0.67$$

Progress

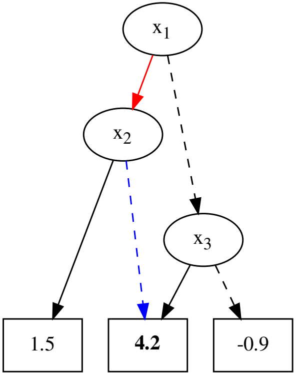
Algebraic Decision Diagram (ADD) [Bahar et al., 1997]

Function $f : \mathbb{B}^3 \to \mathbb{R}$

Exhaustive enumeration

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
T	T	T	1.5
T	T	F	1.5
T	F	T	**4.2**
T	F	F	**4.2**
F	T	T	4.2
F	T	F	-0.9
F	F	T	4.2
F	F	F	-0.9

Algebraic Decision Diagram (ADD)



Variable Eliminations and Function Weight

Arbitrary function:

$$f : \mathbb{B}^2 \to \mathbb{R}$$

Arbitrary real-valued literal weights:

$W(x_1)$	$W(\neg x_1)$
$W(x_2)$	$W(\neg x_2)$

Eliminating variable x_2 from $f(x_1, x_2)$:

$$g = \sum_{x_2}^{W} f \qquad (g : \mathbb{B}^1 \to \mathbb{R})$$

$$g(x_1) = f(x_1, \mathsf{T}) \cdot W(x_2) + f(x_1, \mathsf{F}) \cdot W(\neg x_2)$$

Eliminating variable x_1 from $g(x_1)$:

$$h = \sum_{x_1}^{W} g \qquad (h : \mathbb{B}^0 \to \mathbb{R})$$

$$h() = g(\mathsf{T}) \cdot W(x_1) + g(\mathsf{F}) \cdot W(\neg x_1)$$

Function weight:

$$W(f) = h()$$

Algebraic Decision Diagrams (ADDs) for Model Counting

Boolean formula $\varphi : \mathbb{B}^3 \to \mathbb{B}$

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$

Arbitrary real-valued literal weights:

$$W(x_1), W(x_2), W(x_3)$$
$$W(\neg x_1), W(\neg x_2), W(\neg x_3)$$

Query:

$$W(\varphi) = \ ?$$

Widen codomain of φ to create $f : \mathbb{B}^3 \to \mathbb{R}$

Compute function weight:

$$W(f) = \left(\sum_{x_1}^{W} \sum_{x_2}^{W} \sum_{x_3}^{W} f \right) ()$$

(Represent functions with ADDs)

Computing answer to query:

$$W(\varphi) = W(f)$$

Representations: Monolithic versus Factored

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$

Naive approach: using monolithic representation

- Builds a large ADD for the whole formula φ

$$D(x_1, x_2, x_3) = \texttt{ADD}(\varphi)$$

- Scales poorly for complex formulas
  - $|ADD\_nodes| = O(\exp(|formula\_variables|))$

Our approach: exploiting factored representation

- Builds a small ADD for each clause
- Combines ADDs iteratively
  - Eliminates variables as soon as possible to reduce sizes of ADDs

Exploiting Factored Representation in Conjunctive Normal Form

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$

Build and combine ADDs:

$$D_1(x_1, x_2) = \sum_{x_3} \texttt{ADD}\,(x_1 \vee x_2 \vee \neg x_3)$$

$$D_2(x_1) = \sum_{x_2} \left(D_1(x_1, x_2) \cdot \texttt{ADD}\,(\neg x_2) \cdot \texttt{ADD}\,(x_1 \vee \neg x_2)\right)$$

$$D_3() = \sum_{x_1} \left(D_2(x_1) \cdot \texttt{ADD}\,(x_1)\right)$$

Heuristics:

- Grouping clauses into clusters: $\kappa_1 = \langle x_1 \rangle$, $\kappa_2 = \langle \neg x_2, x_1 \vee \neg x_2 \rangle$, $\kappa_3 = \langle x_1 \vee x_2 \vee \neg x_3 \rangle$
- Combining clusters: $\langle \langle \kappa_3, \kappa_2 \rangle, \kappa_1 \rangle$

Contributions

- Algorithm for weighted model counting using Algebraic Decision Diagrams (ADDs)
  - Exploits factored representation in Conjunctive Normal Form: $\varphi = c_1 \wedge c_2 \wedge \ldots$
    - Builds small clause ADDs then combines them
    - Eliminates variables early
  - Utilizes various heuristics
- Tool: Algebraic Decision Diagram Model Counter (`ADDMC`)

Progress

Comparing Weighted Model Counters

Setup:

- Rice NOTS Linux cluster
- Timeout: 1000-second

Table: 1091 standard weighted model counting benchmarks

Weighted model counter	Benchmarks solved	Percentage solved
`Cachet` [Sang et al., 2004]	776	71%
`miniC2D` [Oztok and Darwiche, 2015]	913	84%
`ADDMC` (our tool)	1085	99%

Comparing Weighted Model Counters

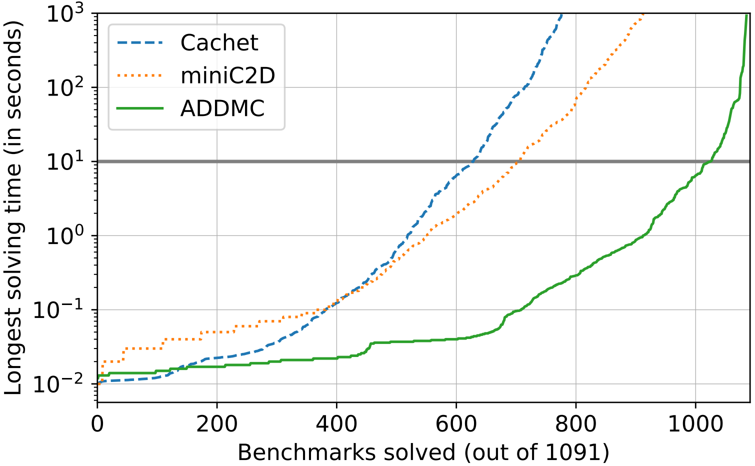


Figure: Cactus plot (rightmost is best)

Conclusion

- Problem: weighted model counting
  - Theoretical hardness: #P-complete
- Our approach: using ADDs, exploiting factored representation
  - Practical efficiency: outperforming other weighted model counters
- Future work:
  - Arbitrary-precision weighted model counting
  - Multi-core computing

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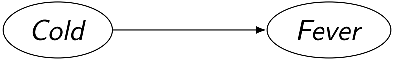
Backup

Backup slides follow

Motivation

Bayesian network:



$P(Cold) = 0.4$

$P(Fever|Cold) = 0.3$
$P(Fever|\neg Cold) = 0.1$

Query: $P(Cold|Fever) = \ ?$

Approaches:

- Variable elimination
- *Recursive conditioning*
- **Reduction to model counting**

Motivation

Real-world diagnostic decision-support tools [Shwe et al., 1991]:

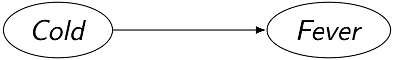
- INTERNIST-1
- Quick Medical Reference (QMR)

Table: QMR-based Bayesian inference benchmarks, with median times in seconds [Sang et al., 2005]

Prior	*Recursive Conditioning* (`SamIam`)	**Reduction to Model counting** (`Cachet`)
0.05	3.5	1.4
0.1	2.5	1.0
0.2	3.4	3.4

From Bayesian Inference to Weighted Model Counting

Bayesian network:



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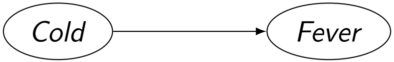
$P(Fever|Cold) = 0.3$
$P(Fever|\neg Cold) = 0.1$

Query: $P(Cold|Fever) = \ ?$

Conversion to model counting instances:

Bayesian element	Variable	Literal weight
$Cold$	x_1	$W(x_1) = 0.4$ $W(\neg x_1) = 0.6$
$P(Fever\|Cold)$	x_2	$W(x_2) = 0.3$ $W(\neg x_2) = 0.7$
$P(Fever\|\neg Cold)$	x_3	$W(x_3) = 0.1$ $W(\neg x_3) = 0.9$
$Fever$	x_4	$W(x_4) = 1$ $W(\neg x_4) = 1$

From Bayesian Inference to Weighted Model Counting



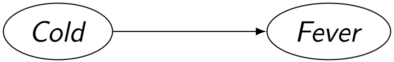
$P(Cold) = 0.4$

$P(Fever|Cold) = 0.3$
$P(Fever|\neg Cold) = 0.1$

Bayesian element	Variable
Cold	x_1
$P(Fever \mid Cold)$	x_2
$P(Fever \mid \neg Cold)$	x_3
Fever	x_4

Bayesian relationship	Implication	Disjunction
$P(Fever \mid Cold) \rightarrow Fever$	$x_2 \rightarrow x_4$	$\neg x_2 \lor x_4$
$Cold \rightarrow (P(Fever \mid Cold) \rightarrow Fever)$	$x_1 \rightarrow (x_2 \rightarrow x_4)$	$\neg x_1 \lor (\neg x_2 \lor x_4)$

From Bayesian Inference to Weighted Model Counting



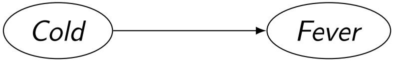
$P(Cold) = 0.4$

$P(Fever|Cold) = 0.3$
$P(Fever|\neg Cold) = 0.1$

Bayesian element	Variable
Cold	x_1
$P(Fever\|Cold)$	x_2
$P(Fever\|\neg Cold)$	x_3
Fever	x_4

Bayesian relationship	Disjunction	Clause
$Cold \rightarrow (P(Fever\|Cold) \rightarrow Fever)$	$\neg x_1 \vee \neg x_2 \vee x_4$	c_1
$Cold \rightarrow (P(Fever\|Cold) \leftarrow Fever)$	$\neg x_1 \vee x_2 \vee \neg x_4$	c_2
$Cold \rightarrow (P(Fever\|\neg Cold) \rightarrow Fever)$	$\neg x_1 \vee \neg x_3 \vee x_4$	c_3
$Cold \rightarrow (P(Fever\|\neg Cold) \leftarrow Fever)$	$\neg x_1 \vee x_3 \vee \neg x_4$	c_4

From Bayesian Inference to Weighted Model Counting



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$P(Fever|\neg Cold) = 0.1$

Bayesian element	Variable
$Cold$	x_1
$P(Fever\|Cold)$	x_2
$P(Fever\|\neg Cold)$	x_3
$Fever$	x_4

Bayesian probability	Conjunction	Formula	Weighted model count
$P(Fever)$	$(c_1 \wedge c_2 \wedge c_3 \wedge c_4) \wedge x_4$	φ	$W(\varphi)$
$P(Fever, Cold)$	$(c_1 \wedge c_2 \wedge c_3 \wedge c_4) \wedge x_4 \wedge x_1$	γ	$W(\gamma)$

$$P(Cold|Fever) = \frac{P(Fever, Cold)}{P(Fever)} = \frac{W(\gamma)}{W(\varphi)} = \ ?$$

From Bayesian Inference to Weighted Model Counting

Clauses:

$$c_1 = \neg x_1 \vee \neg x_2 \vee x_4$$
$$c_2 = \neg x_1 \vee x_2 \vee \neg x_4$$
$$c_3 = x_1 \vee \neg x_3 \vee x_4$$
$$c_4 = x_1 \vee x_3 \vee \neg x_4$$

Formula:

$$\varphi = (c_1 \wedge c_2 \wedge c_3 \wedge c_4) \wedge x_4$$

Model (satisfying assignment) M_1 of φ:

$$M_1 = \{x_1, x_2, x_3, x_4\}$$

Literal weights	
$W(x_1) = 0.4$	$W(\neg x_1) = 0.6$
$W(x_2) = 0.3$	$W(\neg x_2) = 0.7$
$W(x_3) = 0.1$	$W(\neg x_3) = 0.9$
$W(x_4) = 1.0$	$W(\neg x_4) = 1.0$

Weight of model M_1 of φ:

$$\begin{aligned} W(M_1) &= W(x_1) \cdot W(x_2) \cdot W(x_3) \cdot W(x_4) \\ &= 0.012 \end{aligned}$$

From Bayesian Inference to Weighted Model Counting

Models of φ:

$$M_1 = \{x_1, x_2, x_3, x_4\}$$
$$M_2 = \{x_1, x_2, \neg x_3, x_4\}$$
$$M_3 = \{\neg x_1, x_2, x_3, x_4\}$$
$$M_4 = \{\neg x_1, \neg x_2, x_3, x_4\}$$

Literal weights	
$W(x_1) = 0.4$	$W(\neg x_1) = 0.6$
$W(x_2) = 0.3$	$W(\neg x_2) = 0.7$
$W(x_3) = 0.1$	$W(\neg x_3) = 0.9$
$W(x_4) = 1.0$	$W(\neg x_4) = 1.0$

Weighted model count of φ:

$$W(\varphi) = W(M_1) + W(M_2) + W(M_3) + W(M_4) = 0.18$$

Answer to query:

$$P(Cold|Fever) = \frac{W(\gamma)}{W(\varphi)} = \frac{0.12}{0.18} = 0.67$$

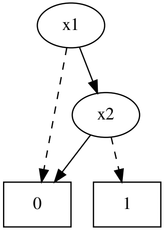
Algebraic Decision Diagrams (ADDs)

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$
$$F_1 = \texttt{ADD}(\varphi)$$

$\varphi : \mathbb{B}^3 \to \mathbb{B}$

x_1	x_2	x_3	$\varphi(x_1, x_2, x_3)$
T	T	T	F
T	T	F	F
T	**F**	**T**	**T**
T	**F**	**F**	**T**
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

$F_1 : \mathbb{B}^3 \to \mathbb{R}$



Algebraic Decision Diagrams (ADDs)

$$\varphi = x_1 \wedge \neg x_2 \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3)$$
$$F_1 = \texttt{ADD}(\varphi)$$

Literal weights	
$W(x_1) = 1.0$	$W(\neg x_1) = 1.0$
$W(x_2) = 0.2$	$W(\neg x_2) = 0.8$
$W(x_3) = 0.3$	$W(\neg x_3) = 0.7$

Query:

$$W(\varphi) = ?$$

Computing answer to query:

$$\begin{aligned} W(\varphi) &= W(F_1) & F_1 &: \mathbb{B}^3 \to \mathbb{R} \\ &= W(F_2) & F_2 &: \mathbb{B}^2 \to \mathbb{R} \\ &= W(F_3) & F_3 &: \mathbb{B}^1 \to \mathbb{R} \\ &= W(F_4) & F_4 &: \mathbb{B}^0 \to \mathbb{R} \end{aligned}$$

Algebraic Decision Diagram (ADD): Variable Elimination

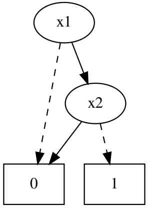
$$F_2(x_1, x_2) = \sum_{x_3} F_1(x_1, x_2, x_3)$$
$$= F_1(x_1, x_2, \mathsf{T}) \cdot W(x_3) + F_1(x_1, x_2, \mathsf{F}) \cdot W(\neg x_3)$$

$$F_1(x_1, x_2, \mathsf{T}) : \mathbb{B}^2 \to \mathbb{R} \quad (1)$$
$$F_1(x_1, x_2, \mathsf{T}) \cdot W(x_3) : \mathbb{B}^2 \to \mathbb{R} \quad (2)$$
$$F_1(x_1, x_2, \mathsf{T}) \cdot W(x_3) + F_1(x_1, x_2, \mathsf{F}) \cdot W(\neg x_3) : \mathbb{B}^2 \to \mathbb{R} \quad (3)$$

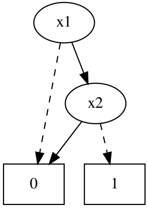
Algebraic Decision Diagram (ADD): Variable Elimination

$$F_2(x_1, x_2) = \sum_{x_3} F_1(x_1, x_2, x_3)$$

ADD $F_1 : \mathbb{B}^3 \to \mathbb{R}$



ADD $F_2 : \mathbb{B}^2 \to \mathbb{R}$

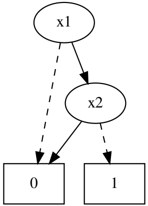


$$W(F_1) = W(F_2)$$

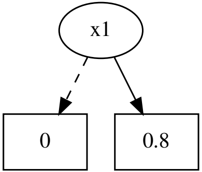
Algebraic Decision Diagram (ADD): Variable Elimination

$$F_3(x_1) = \sum_{x_2} F_2(x_1, x_2)$$

ADD $F_2 : \mathbb{B}^2 \to \mathbb{R}$



ADD $F_3 : \mathbb{B}^1 \to \mathbb{R}$

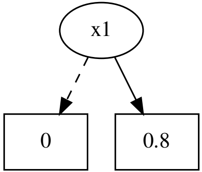


$$W(F_2) = W(F_3)$$

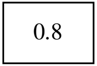
Algebraic Decision Diagram (ADD): Variable Elimination

$$F_4() = \sum_{x_1} F_3(x_1)$$

ADD $F_3 : \mathbb{B}^1 \to \mathbb{R}$



ADD $F_4 : \mathbb{B}^0 \to \mathbb{R}$



$$W(F_3) = W(F_4) = 0.8 = W(\varphi)$$

Algebraic Decision Diagrams (ADDs) for Model Counting

$$\varphi = x_1 \land \neg x_2 \land (x_1 \lor \neg x_2) \land (x_1 \lor x_2 \lor \neg x_3)$$

Literal weights	
$W(x_1) = 1.0$	$W(\neg x_1) = 1.0$
$W(x_2) = 0.2$	$W(\neg x_2) = 0.8$
$W(x_3) = 0.3$	$W(\neg x_3) = 0.7$

$$F_4() = \sum_{x_1} \sum_{x_2} \sum_{x_3} \texttt{ADD}(\varphi)$$

$$W(F_4) = 0.8 = W(\varphi)$$